

TRACKING CHARGED
PARTICLES IN THE
ZERO CURVATURE
LIMIT

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**TRACKING CHARGED
PARTICLES IN THE
ZERO CURVATURE
LIMIT**

Alcaraz Maestre, J.

TRAZADO DE PARTÍCULAS CARGADAS EN EL LÍMITE DE CURVATURA NULA

Alcaraz Maestre, J.

24 pp, 14 refs , 5 figs

Resumen:

En este informe discutimos estrategias apropiadas para el trazado de partículas cargadas en el límite de curvatura nula. Los procedimientos sugeridos evitan tratamientos especiales y problemas de precisión que aparecen frecuentemente en ese límite. Proporcionamos asimismo expresiones explícitas para el transporte, re-ajuste y cálculo de vértices en regiones donde las inhomogeneidades del campo magnético o efectos de interacción con el detector puedan ser aproximadamente ignorados.

TRACKING CHARGED PARTICLES IN THE ZERO CURVATURE LIMIT

Alcaraz Maestre, J.

24 pp, 14 refs , 5 figs

Abstract:

In this report we discuss appropriate strategies for the tracking of charged particles in the limit of zero curvature. The suggested approach avoids special treatments and precision issues that frequently arise in that limit. We provide explicit expressions for transport, refitting and vertexing in regions where magnetic field inhomogeneities or detector interaction effects can be approximately ignored.

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1 INTRODUCTION

The linear relation between the transverse momentum p_T of a charged particle and the inverse of its curvature C in the presence of a magnetic field constitutes the basis for the determination of momenta in particle physics. This relation is typically exploited by tracking and analyzing the ionization deposits along the trajectory of the particle. Sophisticated tracking techniques in past and present experiments have resulted into major physical results and discoveries. Getting sufficiently precise measurements will become particularly challenging at future colliders with multi-TeV center-of-mass energies, due to the much lower expected curvatures. In this context, ultra-precise detectors with $\lesssim 10 \mu\text{m}$ of position and alignment accuracies over distances of meters, able to support high occupancies of hundreds or even thousands of tracks per event will be required.

The zero curvature limit ($C \rightarrow 0$) or, equivalently, the transverse momentum limit ($p_T \rightarrow \infty$) is always a potential source of problems for tracking algorithms, due to presence of divergent operations in many of the used equations. A dangerous approach is to consider $C = 0$ as a special, exceptional case. The problem occurs in practice when the curvature gets smaller or comparable to the computer precision used in the calculations, C_{prec} . Apart from the inherent complications in the code, the scheme is intrinsically limited because it abandons by construction the possibility of providing sensible measurements for tracks with $C < C_{prec}$.

This report discusses tracking in the zero curvature limit, based on some past notes written in the running period of the L3 detector at LEP [1]. It adopts the basic strategy of avoiding the use of any curvature threshold in a systematic way. For that, we work at all times with relatively simple expressions that do not present divergences in the $C \rightarrow 0$ limit. This turns out to be optimal to obtain precise measurements at high momenta. The resulting equations were successfully employed already in past tracking and detector alignment algorithms, starting with the L3 detector at LEP [1–4].

Only tracking in homogeneous magnetic fields is considered, but one has to keep in mind that all inhomogeneous cases are decomposed in practice into a series of small propagation steps where the present formulae are always applicable. Material effects, which are more frequently discussed in connection with filtering techniques [5], do not have an influence on the choice of the parametrization near the collision point. Specific approaches to the case of slightly inhomogeneous magnetic fields are discussed for instance in References [6, 7].

The document is organized as follows. Section 2 introduces the “perigee” parametrization that will be used throughout most of the report. It also connects it with other equivalent conventions used in the literature. The propagation of track parameters and related uncertainties is discussed in sections 3, 4, including specific subsections for the cases of propagation to cylinder and plane surfaces (sections 4.4 and 4.5). Sections 5 and 5.3 address the typical problem of track refitting when new measurements are present, which is of utmost importance in alignment, fitting and filtering techniques in general. Finally, sections 6 and 7 discuss vertex determination in a simple but precise way.

2 CONVENTIONS

A helix is the most general trajectory followed by a charged particle in a homogeneous, constant in time, magnetic field region in vacuum. We assume that the direction of the magnetic field coincides with the positive direction of the Z axis in the reference system. The trajectory can thus be visualized in XY as a circumference (Figure 1). The Z displacement between two positions in the trajectory is proportional to the arc length that is described in the XY plane, leading to the concept of a “straight line in the SZ plane”, where S identifies the variable associated to the arc length (Figure 2). Unless otherwise stated, equations assume that distances are given in meters, magnetic field strengths in Teslas, charge in units of the positron charge and momenta in GeV/c. We will consider a standard decomposition of the local momentum vector, \hat{p} , in spherical coordinates: $\hat{p} \equiv (p \sin \theta \cos \phi, p \sin \theta \sin \phi, p \cos \theta)$, where p is its magnitude and ϕ and θ are the associated azimuthal and polar angles. Since the magnetic field is homogeneous and does not change with time, p and θ are constants of motion. The angles are chosen such that: $\phi \in (-\pi, \pi]$ and $\theta \in [0, \pi]$.

In the XY plane the trajectory can be fully described by a reference point, (x_r, y_r) , and three parameters, C , ϕ_0 and δ (Figure 1):

- C : the curvature of the track. It is related with the absolute value of the transverse momentum p_T via the Equation: $C = 0.29979 \text{ Bq}/p_T$, where B is the magnetic field strength and q the particle charge. The curvature is positive (negative) if the particle has a positive (negative) charge. A positive (negative) curvature identifies a clockwise (anti-clockwise) rotation of the particle in the XY coordinate system as a function of time.
- ϕ_0 : the azimuthal angle of the momentum vector at the position of closest approach to the reference point.
- δ : the distance of closest approach to the reference point. This is a signed parameter, with a convention such that the coordinates of closest approach (x_0, y_0) are given by:

$$\left. \begin{array}{l} x_0 = x_r - \delta \sin \phi_0 \\ y_0 = y_r + \delta \cos \phi_0 \end{array} \right\} \Rightarrow \delta = -(x_0 - x_r) \sin \phi_0 + (y_0 - y_r) \cos \phi_0 \quad (\text{Eq. 1})$$

The trajectory in the SZ plane is simply described by a straight line:

$$z = z_0 + s \tan \lambda \quad (\text{Eq. 2})$$

where s is the arc length traversed when the particle moves from (x_0, y_0) to (x, y) and the parameters $\tan \lambda$ and z_0 have a simple interpretation (Figure 2):

- $\tan \lambda$: the slope in the SZ plane, dz/ds . It is directly related with the polar angle θ by: $\tan \lambda = 1/\tan \theta$.

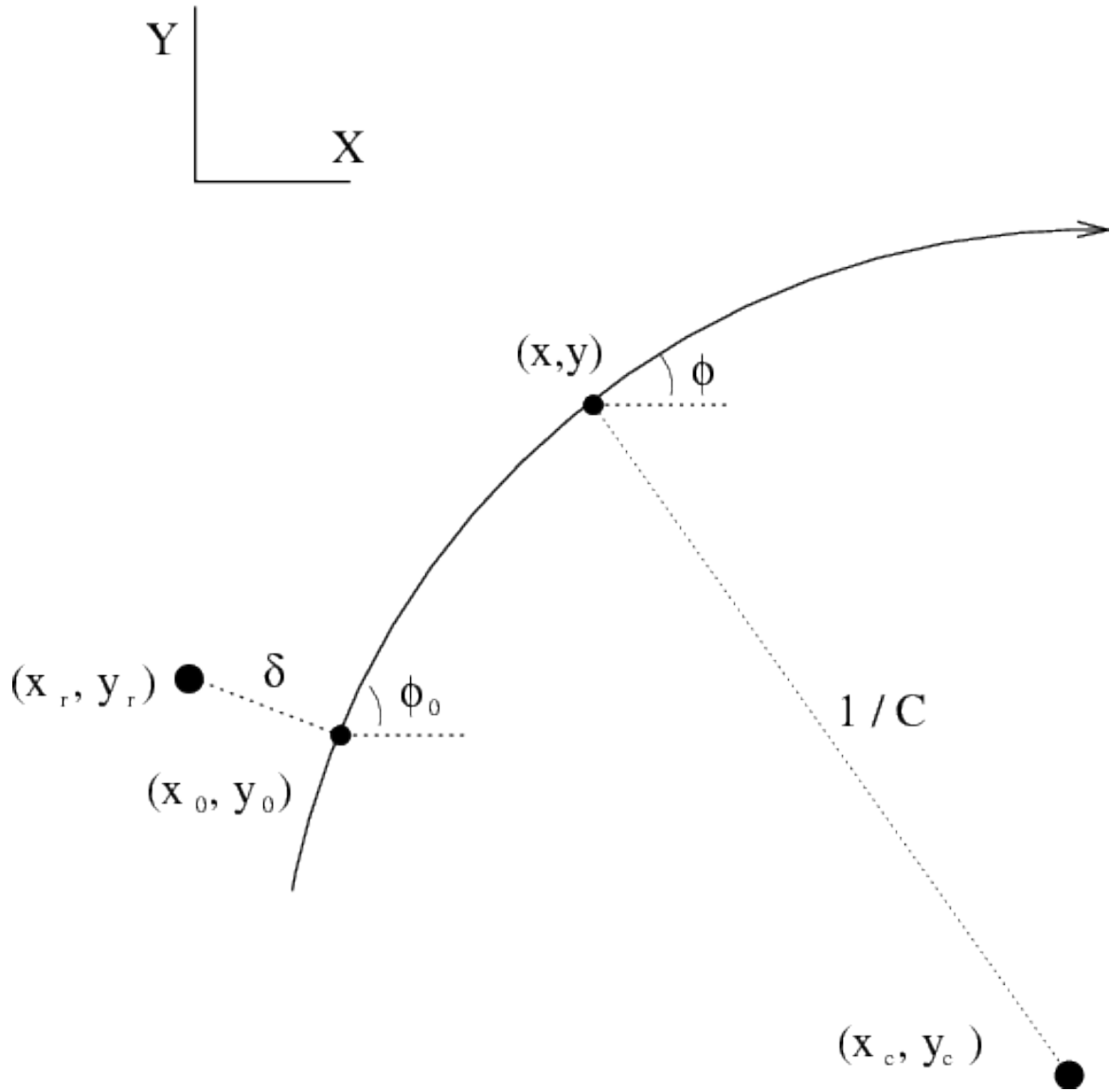


Figure 1. The projection of the track helix on the XY plane will be a circumference. The parameters shown in the figure are described in the text.

- z_0 : the Z position when the particle is at the distance of closest approach in the XY plane.

Let us note that the arc length s can be either positive or negative, depending on whether (x, y) corresponds to a position occurring before or after (x_0, y_0) in time.

In summary, in this “perigee” convention, a track is fully defined by a reference point (x_r, y_r) and the set of parameters $(C, \phi_0, \delta, z_0, \tan \lambda)$. This is the convention used for instance by the CMS [8] and L3 [2] experiments to describe track parameters and uncertainties close to the interaction point ¹. According to the previous discussion, C and $\tan \lambda$ are constant over the trajectory.

¹The L3 parametrization was only differing by a change in the sign convention for the δ parameter.

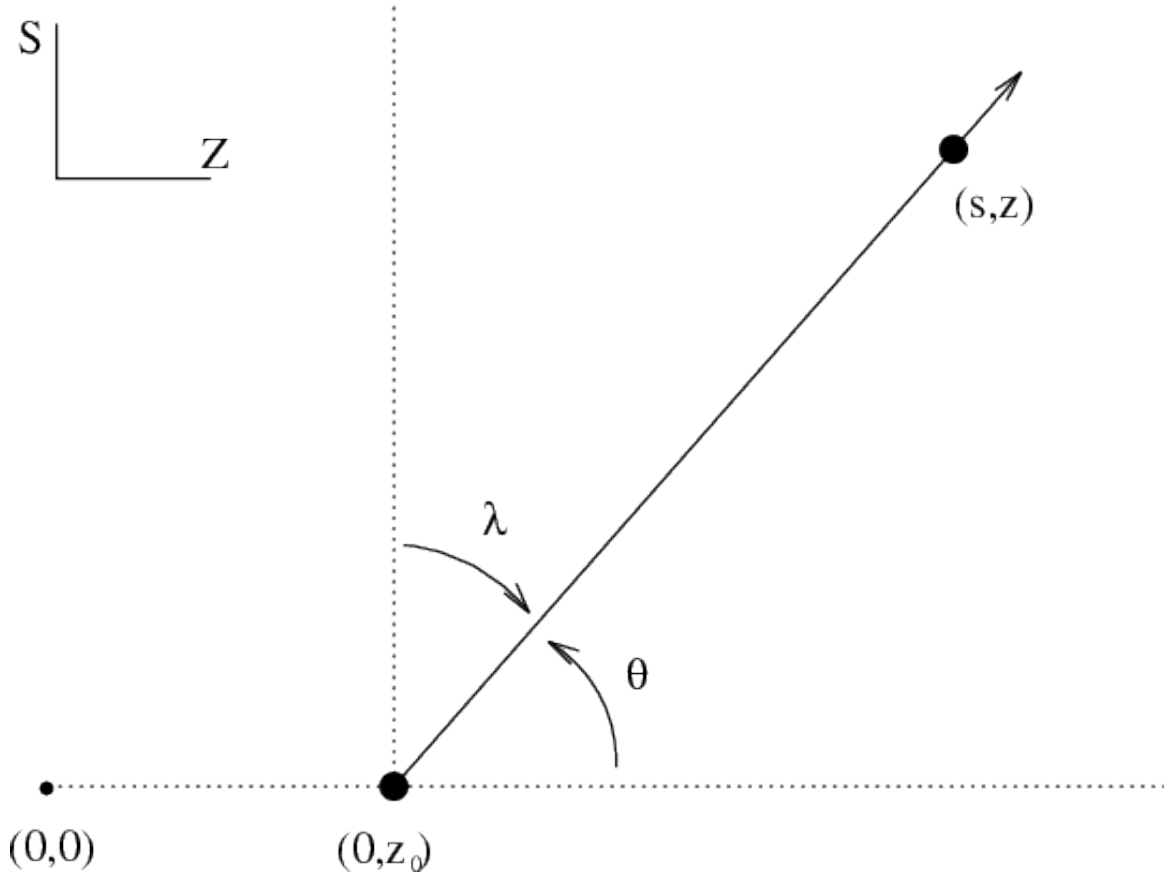


Figure 2. Following the adopted convention, the projection of the track on the SZ plane will be a straight line: $z = z_0 + s \tan \lambda$. The relevant parameters are shown in the Figure. Note that the variable s at a point (x, y, z) is equal to the arc length in the XY plane between (x_0, y_0) to (x, y) . This also implies that $s = 0$ at $z = z_0$.

2.1 ALTERNATIVE PARAMETRIZATIONS

There are equivalent parametrizations in the literature, already used in past or existing high-energy physics experiments. They differ slightly from the one employed in this report:

- An alternative perigee parametrization [9], rather similar to our choice. The most relevant difference is the use of the parameter θ instead of $\tan \lambda$. Both parameters are constant in the case of homogeneous fields, but $\tan \lambda$ is the slope of the straight line fit in the SZ plane, a technically natural coefficient for detectors embedded in a solenoidal field. Explicitly, this alternative choice employs the parameters ρ , ϕ_p , ϵ , θ and z_p , which are connected with our parameters through the relations: $\rho = -C$, $\phi_p = \phi_0$, $\epsilon = -\delta$, $\cot \theta = \tan \lambda$, $z_p = z_0$.
- A variation of the previous “perigee” parametrization [9] is the one described for instance in Reference [10], which is used in ATLAS [11]. It basically substitutes the curvature C by the inverse of the transverse rigidity, q/p_T . The relation between the curvature and q/p_T is: $q/p_T = (C \cos \lambda)/(0.29979 \text{ B})$. Note that this parametrization requires the a priori knowledge of the magnetic field strength, something that is not intrinsically necessary to reconstruct tracks in regions with homogeneous magnetic fields and negligible interactions with the medium.
- The curvilinear parametrization [12] adopts a local description of the track with the parameters

$(q/p, \phi, x_{\perp}, \lambda, z_{\perp})$. Here x_{\perp} and z_{\perp} quantify displacements in a plane transverse to the trajectory. When the point on the track is the one with closest transverse approach with respect to (x_r, y_r) : $\phi = \phi_0$, $x_{\perp} = \delta$ and $z_{\perp} = z_0 \cos \lambda$.

All equations presented in this report can be adapted to any of these alternative parametrizations in a straightforward way. For completeness, we also collect in Appendix A the Jacobians of the transformations to the alternative conventions. These Jacobians are required for the propagation of the uncertainties discussed in Section 4.

3 SOME USEFUL FORMULAE FOR PROPAGATION ALGORITHMS IN THE ZERO-CURVATURE LIMITS

Most of the problems encountered in propagation algorithms admit several solutions from a formal point of view. However, many of these solutions are not optimal, neither from the point of view of precision, nor from the point of view of simplicity. The relations shown below are free of divergences in the $C \rightarrow 0$ limit.

3.1 DETERMINING THE AZIMUTHAL ANGLE OF THE TRAJECTORY AT A GIVEN POINT IN THE TRANSVERSE PLANE.

If the curvature is already available, an optimal expression to determine ϕ is the following:

$$\phi = \text{atan2}(\sin \phi_0 - C(x - x_0), \cos \phi_0 + C(y - y_0)) \quad (\text{Eq. 3})$$

where $\text{atan2}(y, x)$ is the function that returns the azimuthal angle of the vector (x, y) in the range $(-\pi, \pi]$. If one prefers not to use C , an alternative expression is:

$$\phi = \text{InRange}(2 \text{atan2}(y - y_0, x - x_0) - \phi_0) \quad (\text{Eq. 4})$$

where $\text{InRange}(\phi)$ is the function that brings the angle ϕ into the $(-\pi, \pi]$ range:

$$\text{InRange}(\phi) = \begin{cases} \phi + 2\pi \text{int}(0.5 - \frac{\phi}{2\pi}) & , \text{ for } \phi \leq \pi \\ \phi - 2\pi \text{int}(0.5 + \frac{\phi}{2\pi}) & , \text{ for } \phi > \pi \end{cases} \quad (\text{Eq. 5})$$

Here $\text{int}(x)$ is the integer part of the real number x . Both Equations, Eq. 3 and Eq. 4, give a precise answer in the $C \rightarrow 0$ limit.

3.2 CURVATURE WHEN ONLY TWO POINTS IN THE TRANSVERSE PLANE AND THE AZIMUTHAL ANGLE AT ONE OF THESE POINTS ARE KNOWN.

The optimal and shortest answer for this problem is:

$$C = \frac{2(x - x_0) \sin \phi_0 - 2(y - y_0) \cos \phi_0}{(x - x_0)^2 + (y - y_0)^2} \quad (\text{Eq. 6})$$

where ϕ_0 is the azimuthal angle at the point (x_0, y_0) and (x, y) is a second point on the trajectory.

3.3 ARC LENGTH AT A GIVEN POINT OF THE TRAJECTORY.

If we define the following projected components of the distance between the point of closest transverse approach (x_0, y_0) and the given point (x, y) :

$$\Delta_{\parallel} = (x - x_0) \cos \phi_0 + (y - y_0) \sin \phi_0 \quad (\text{Eq. 7})$$

$$\Delta_{\perp} = -(x - x_0) \sin \phi_0 + (y - y_0) \cos \phi_0, \quad (\text{Eq. 8})$$

we obtain the solution, well behaved in the zero curvature limit:

$$s = \frac{\text{atan2}(C \Delta_{\perp}, 1 + C \Delta_{\parallel})}{C} \quad (\text{Eq. 9})$$

If we have access to the local azimuthal angle ϕ at (x, y) (see also subsection 3.1), we can use instead a relation that is totally independent of C :

$$s = \frac{\Delta_{\parallel}}{\text{sinc}(\phi - \phi_0)} \quad (\text{Eq. 10})$$

where $\text{sinc}(x)$ is the function $(\sin x)/x$.

3.4 DETERMINING THE POSITION ON THE TRAJECTORY FOR A GIVEN ARC LENGTH.

To avoid accuracy problems one should use the curvature C in a special way:

$$x = x_0 + s \text{sinc}\left(\frac{Cs}{2}\right) \cos\left(\phi_0 - \frac{Cs}{2}\right) \quad (\text{Eq. 11})$$

$$y = y_0 + s \text{sinc}\left(\frac{Cs}{2}\right) \sin\left(\phi_0 - \frac{Cs}{2}\right), \quad (\text{Eq. 12})$$

If ϕ is available, one can make the substitution: $Cs = \phi_0 - \phi$, and use instead:

$$x = x_0 - s \text{sinc}\left(\frac{\phi - \phi_0}{2}\right) \cos\left(\frac{\phi + \phi_0}{2}\right) \quad (\text{Eq. 13})$$

$$y = y_0 - s \text{sinc}\left(\frac{\phi - \phi_0}{2}\right) \sin\left(\frac{\phi + \phi_0}{2}\right), \quad (\text{Eq. 14})$$

4 PROPAGATION OF TRACK PARAMETERS AND UNCERTAINTIES

In this section we provide expressions to update the track parameters and the associated covariance matrix when a new reference point (x'_r, y'_r, z'_r) , different from the original one (x_r, y_r, z_r) , is used in the parametrization (Figure 3). There are two important use cases that are covered with this exercise: a) a track referred to a new point that is different from the initial estimate used in reconstruction (typically the beam spot); b) the propagation to the crossing point between the track and any given subdetector surface.

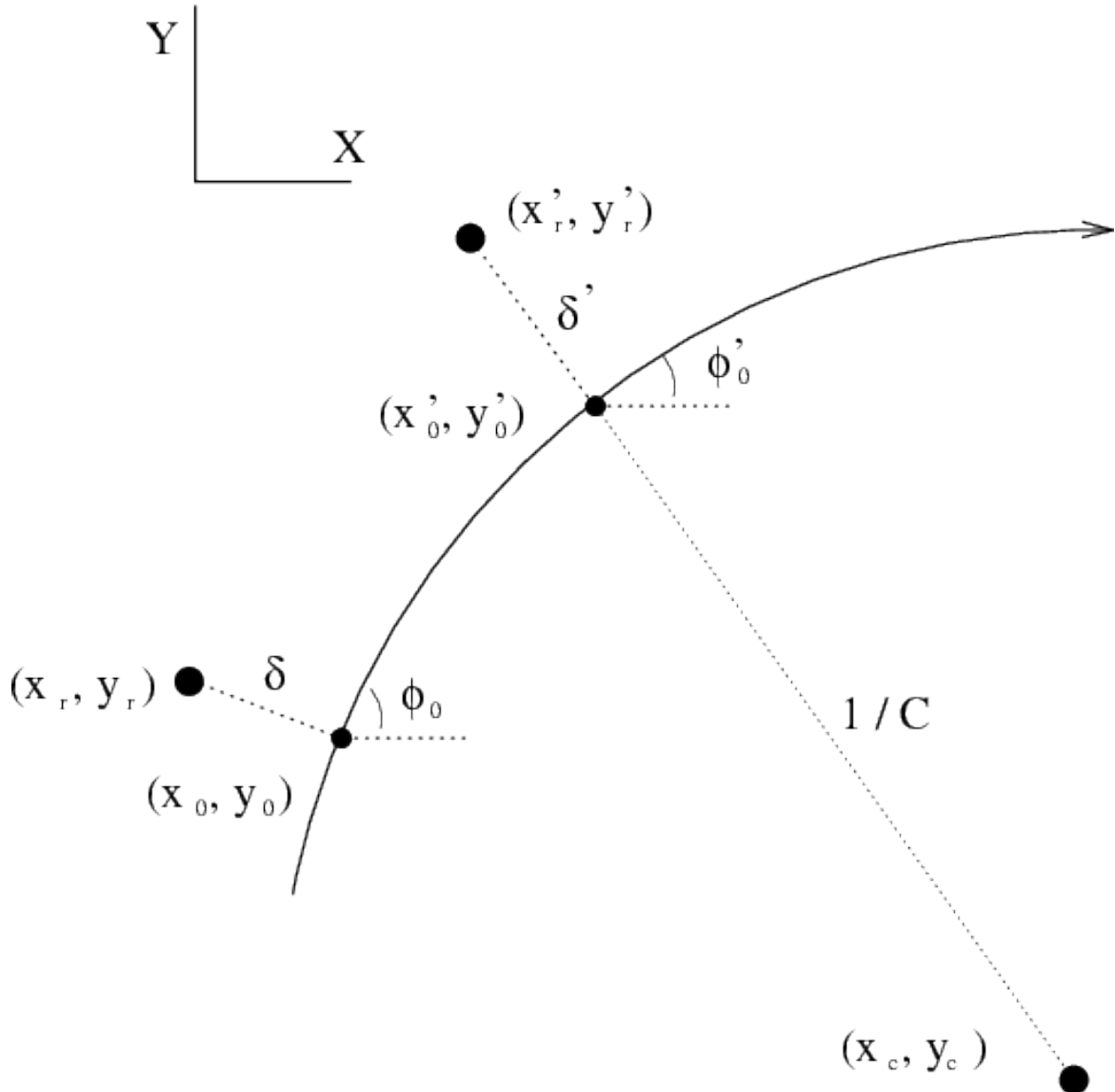


Figure 3. Change of parameters in XY when the reference point moves from (x_r, y_r) to (x'_r, y'_r) .

4.1 UPDATED TRACK PARAMETERS AT A NEW REFERENCE POINT

Changing the reference point of a track implies a modification of its parameters ϕ_0 , δ and z_0 . We will denote the new parameters by ϕ'_0 , δ' and z'_0 . Since $\text{sign}(1 - C\delta) = \text{sign}(1 - C\delta')$ for all cases of

practical interest, ϕ'_0 can be unambiguously determined as follows:

$$\phi'_0 = \text{atan2} \left(\sin \phi_0 - \frac{C\Delta_x}{1 - C\delta}, \cos \phi_0 + \frac{C\Delta_y}{1 - C\delta} \right) \quad (\text{Eq. 15})$$

where $\Delta_x = x'_r - x_r$ and $\Delta_y = y'_r - y_r$. Once ϕ'_0 is available, a convenient expression to determine δ' , also with a good behavior in the $C \rightarrow 0$ limit, is:

$$\delta' = \delta + \Delta_x \sin \phi_0 - \Delta_y \cos \phi_0 + (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \tan\left(\frac{\phi'_0 - \phi_0}{2}\right) \quad (\text{Eq. 16})$$

The point of minimum transverse approach to (x'_r, y'_r) is then simply given by:

$$x'_0 = x'_r - \delta' \sin \phi'_0 \quad (\text{Eq. 17})$$

$$y'_0 = y'_r + \delta' \cos \phi'_0, \quad (\text{Eq. 18})$$

leading to the following expression for the determination of the z'_0 parameter:

$$z'_0 = z_0 + \frac{(x'_0 - x_0) \cos \phi_0 + (y'_0 - y_0) \sin \phi_0}{\text{sinc}(\phi'_0 - \phi_0)} \tan \lambda \quad (\text{Eq. 19})$$

4.2 APPROXIMATE SOLUTIONS FOR THE DISTANCE OF CLOSEST APPROACH TO A GIVEN POINT

Equation Eq. 16 can be well approximated in the $C\delta' \ll 1$ limit, i.e. when the new reference point is very close to the particle trajectory:

$$\delta' \approx \delta - \frac{C\delta^2}{2} + (\Delta_x \sin \phi_0 - \Delta_y \cos \phi_0) (1 - C\delta) - C \frac{\Delta_x^2 + \Delta_y^2}{2} \quad (\text{Eq. 20})$$

where the neglected term on the right hand side is $C\delta'^2/2$. The expression can be further simplified if $C\delta^2$ and $C\delta\delta'$ terms are also negligible:

$$\delta' \approx \delta + \Delta_x \sin \phi_0 - \Delta_y \cos \phi_0 - \frac{C}{2(1 - C\delta)} (\Delta_x^2 + \Delta_y^2) \quad (\text{Eq. 21})$$

This approximation can be used as the starting point for the determination of the track parameters C, ϕ_0, δ [13]. It is indeed one of the preferred methods when energy losses are not important, and it was successfully employed in several past high-energy physics experiments.

4.3 UPDATED COVARIANCE MATRIX AFTER PROPAGATION TO A NEW POINT OR SURFACE

The new covariance matrix after propagation, V' , can be determined in a linear approximation as:

$$V' = J \left(\frac{\partial(C', \phi'_0, \delta', \tan \lambda', z'_0)}{\partial(C, \phi_0, \delta, \tan \lambda, z_0)} \right) V J^T \left(\frac{\partial(C', \phi'_0, \delta', \tan \lambda', z'_0)}{\partial(C, \phi_0, \delta, \tan \lambda, z_0)} \right)^T \quad (\text{Eq. 22})$$

where J is the Jacobian matrix of the transportation, J^T its transpose and V is the original covariance matrix. We will implicitly assume that the full covariance matrix can be built separately in the XY and SZ projections. This is a good approximation if: a) no significant correlations exist between measurements in the XY and SZ planes or b) measurements in the XY plane are much more precise than measurements in the SZ plane. Both conditions are generally satisfied in high-energy particle detectors. Condition b) is satisfied by design in most cases of interest because the best precision is always required in the plane that measures the transverse momentum, i.e. XY. Condition a) is satisfied when active detectors adopt an axial geometry around accelerator beams. An exception is the case of silicon microstrip detectors with slightly tilted stereo-layers². In this case, condition b) is approximately satisfied, the influence of SZ parameter uncertainties on the XY fit can be neglected and the arc lengths described in the XY plane can be considered as extremely precise compared with Z uncertainties in the SZ straight line fit.

The Jacobian matrix can be therefore expressed as:

$$J \left(\frac{\partial(C', \phi'_0, \delta', \tan \lambda', z'_0)}{\partial(C, \phi_0, \delta, \tan \lambda, z_0)} \right) \equiv \left(\begin{array}{ccc|cc} \frac{\partial C'}{\partial C} & \frac{\partial C'}{\partial \phi_0} & \frac{\partial C'}{\partial \delta} & 0 & 0 \\ \frac{\partial \phi'_0}{\partial C} & \frac{\partial \phi'_0}{\partial \phi_0} & \frac{\partial \phi'_0}{\partial \delta} & 0 & 0 \\ \frac{\partial \delta'}{\partial C} & \frac{\partial \delta'}{\partial \phi_0} & \frac{\partial \delta'}{\partial \delta} & 0 & 0 \\ \frac{\partial \tan \lambda'}{\partial C} & \frac{\partial \tan \lambda'}{\partial \phi_0} & \frac{\partial \tan \lambda'}{\partial \delta} & \frac{\partial \tan \lambda'}{\partial \tan \lambda} & \frac{\partial \tan \lambda'}{\partial z_0} \\ 0 & 0 & 0 & \frac{\partial z'_0}{\partial \tan \lambda} & \frac{\partial z'_0}{\partial z_0} \\ 0 & 0 & 0 & \frac{\partial z'_0}{\partial \tan \lambda} & \frac{\partial z'_0}{\partial z_0} \end{array} \right) \quad (\text{Eq. 23})$$

After careful processing of all terms with a potentially bad behavior in the straight line limit we obtain for the XY sub-matrix associated to XY measurements:

$$\frac{\partial C'}{\partial C} = 1 \quad (\text{Eq. 24})$$

$$\frac{\partial C'}{\partial \phi_0} = 0 \quad (\text{Eq. 25})$$

$$\frac{\partial C'}{\partial \delta} = 0 \quad (\text{Eq. 26})$$

²This tilt corresponds to a small rotation around the $(\cos \phi_d, \sin \phi_d, 0)$ direction, where ϕ_d is the azimuthal angle of the center of the layer at $z = 0$.

$$\frac{\partial \phi'_0}{\partial C} = -\frac{1}{(1 - C\delta')^2} (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \quad (\text{Eq. 27})$$

$$\frac{\partial \phi'_0}{\partial \phi_0} = \left(\frac{1 - C\delta}{1 - C\delta'} \right) + \frac{C^2(1 - C\delta)}{2(1 - C\delta')^2} (\delta'^2 - (\Delta_x + \delta \sin \phi_0)^2 - (\Delta_y - \delta \cos \phi_0)^2) \quad (\text{Eq. 28})$$

$$\frac{\partial \phi'_0}{\partial \delta} = -\frac{C^2}{(1 - C\delta')^2} (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \quad (\text{Eq. 29})$$

$$\frac{\partial \delta'}{\partial C} = \frac{1}{2(1 - C\delta')} (\delta'^2 - (\Delta_x + \delta \sin \phi_0)^2 - (\Delta_y - \delta \cos \phi_0)^2) \quad (\text{Eq. 30})$$

$$\frac{\partial \delta'}{\partial \phi_0} = \frac{1 - C\delta}{1 - C\delta'} (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \quad (\text{Eq. 31})$$

$$\frac{\partial \delta'}{\partial \delta} = 1 + \frac{C^2}{2(1 - C\delta')} (\delta'^2 - (\Delta_x + \delta \sin \phi_0)^2 - (\Delta_y - \delta \cos \phi_0)^2) \quad (\text{Eq. 32})$$

with a smooth transition to the straight-line track limit:

$$\frac{\partial C'}{\partial C} = 1 \quad (\text{Eq. 33})$$

$$\frac{\partial C'}{\partial \phi_0} = 0 \quad (\text{Eq. 34})$$

$$\frac{\partial C'}{\partial \delta} = 0 \quad (\text{Eq. 35})$$

$$\frac{\partial \phi'_0}{\partial C} = -(\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \quad (\text{Eq. 36})$$

$$\frac{\partial \phi'_0}{\partial \phi_0} = 1 \quad (\text{Eq. 37})$$

$$\frac{\partial \phi'_0}{\partial \delta} = 0 \quad (\text{Eq. 38})$$

$$\frac{\partial \delta'}{\partial C} = \frac{1}{2} (\delta'^2 - (\Delta_x + \delta \sin \phi_0)^2 - (\Delta_y - \delta \cos \phi_0)^2) \quad (\text{Eq. 39})$$

$$\frac{\partial \delta'}{\partial \phi_0} = (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \quad (\text{Eq. 40})$$

$$\frac{\partial \delta'}{\partial \delta} = 1 \quad (\text{Eq. 41})$$

The derivatives required to build the $(\tan \lambda, z_0)$ covariance sub-matrix are simpler:

$$\frac{\partial \tan \lambda'}{\partial \tan \lambda} = 1 \quad (\text{Eq. 42})$$

$$\frac{\partial \tan \lambda'}{\partial z_0} = 0 \quad (\text{Eq. 43})$$

$$\frac{\partial z'_0}{\partial \tan \lambda} = s \quad (\text{Eq. 44})$$

$$\frac{\partial z'_0}{\partial z_0} = 1 \quad (\text{Eq. 45})$$

For other track parametrizations (see section 2.1), an additional conversion will be necessary: $V'_{par} =$

$J_{par} V_{par} J_{par}^T$, where V_{par} is the covariance matrix of the alternative parametrization and J_{par} is related with the Jacobian matrices given in Appendix A via the formula:

$$J_{par} = J \left(\frac{\partial(\text{new par.})}{\partial(C, \phi_0, \delta, \dots)} \right) J \left(\frac{\partial(C', \phi'_0, \delta', \dots)}{\partial(C, \phi_0, \delta, \dots)} \right) J \left(\frac{\partial(\text{new par.})}{\partial(C, \phi_0, \delta, \dots)} \right)^{-1} \quad (\text{Eq. 46})$$

4.4 TRACK CROSSING A CYLINDER WITH AXIS PARALLEL TO Z

The case is illustrated in Figure 4. The crossing point (x'_0, y'_0, z'_0) is defined by the conditions:

$$C x'_0 + \sin \phi'_0 = C x_0 + \sin \phi_0 \quad (\text{Eq. 47})$$

$$C y'_0 - \cos \phi'_0 = C y_0 - \cos \phi_0 \quad (\text{Eq. 48})$$

$$z'_0 = z_0 + s \tan \lambda \quad (\text{Eq. 49})$$

$$\rho^2 = (x'_0 - x_\rho)^2 + (y'_0 - y_\rho)^2 \quad (\text{Eq. 50})$$

where (x_ρ, y_ρ) is the center of the circumference being crossed in the XY plane and ρ is its radius. To solve the system of equations we use the following definitions:

$$\phi_\rho = \text{atan2}(\sin \phi_0 - C(x_\rho - x_0), \cos \phi_0 + C(y_\rho - y_0)) \quad (\text{Eq. 51})$$

$$\phi_c = \text{atan2}(y'_0 - y_\rho, x'_0 - x_\rho) \quad (\text{Eq. 52})$$

$$\gamma = \frac{2(x_\rho - x_0) \sin \phi_0 - 2(y_\rho - y_0) \cos \phi_0 - C\rho^2 - C((x_\rho - x_0)^2 + (y_\rho - y_0)^2)}{2\rho\sqrt{(\sin \phi_0 - C(x_\rho - x_0))^2 + (\cos \phi_0 + C(y_\rho - y_0))^2}} \quad (\text{Eq. 53})$$

The implicit solution is simply:

$$\sin(\phi_c - \phi_\rho) = \gamma \quad (\text{Eq. 54})$$

which has a meaning only if $|\gamma| < 1$ and provides two possible solutions for ϕ_c (as visually expected). In terms of ϕ_c , the values of x'_0, y'_0, ϕ'_0 and z'_0 are given by:

$$x'_0 = x_\rho + \rho \cos \phi_c \quad (\text{Eq. 55})$$

$$y'_0 = y_\rho + \rho \sin \phi_c \quad (\text{Eq. 56})$$

$$\phi'_0 = \text{atan2}(\sin \phi_0 - C(x'_0 - x_0), \cos \phi_0 + C(y'_0 - y_0)) \quad (\text{Eq. 57})$$

$$z'_0 = z_0 + \left[\frac{(x'_0 - x_0) \cos \phi_0 + (y'_0 - y_0) \sin \phi_0}{\text{sinc}(\phi'_0 - \phi_0)} \right] \tan \lambda \quad (\text{Eq. 58})$$

In most cases we will only be interested in solutions with a positive arc length:

$$(x'_0 - x_0) \cos \phi_0 + (y'_0 - y_0) \sin \phi_0 > 0 \quad (\text{Eq. 59})$$

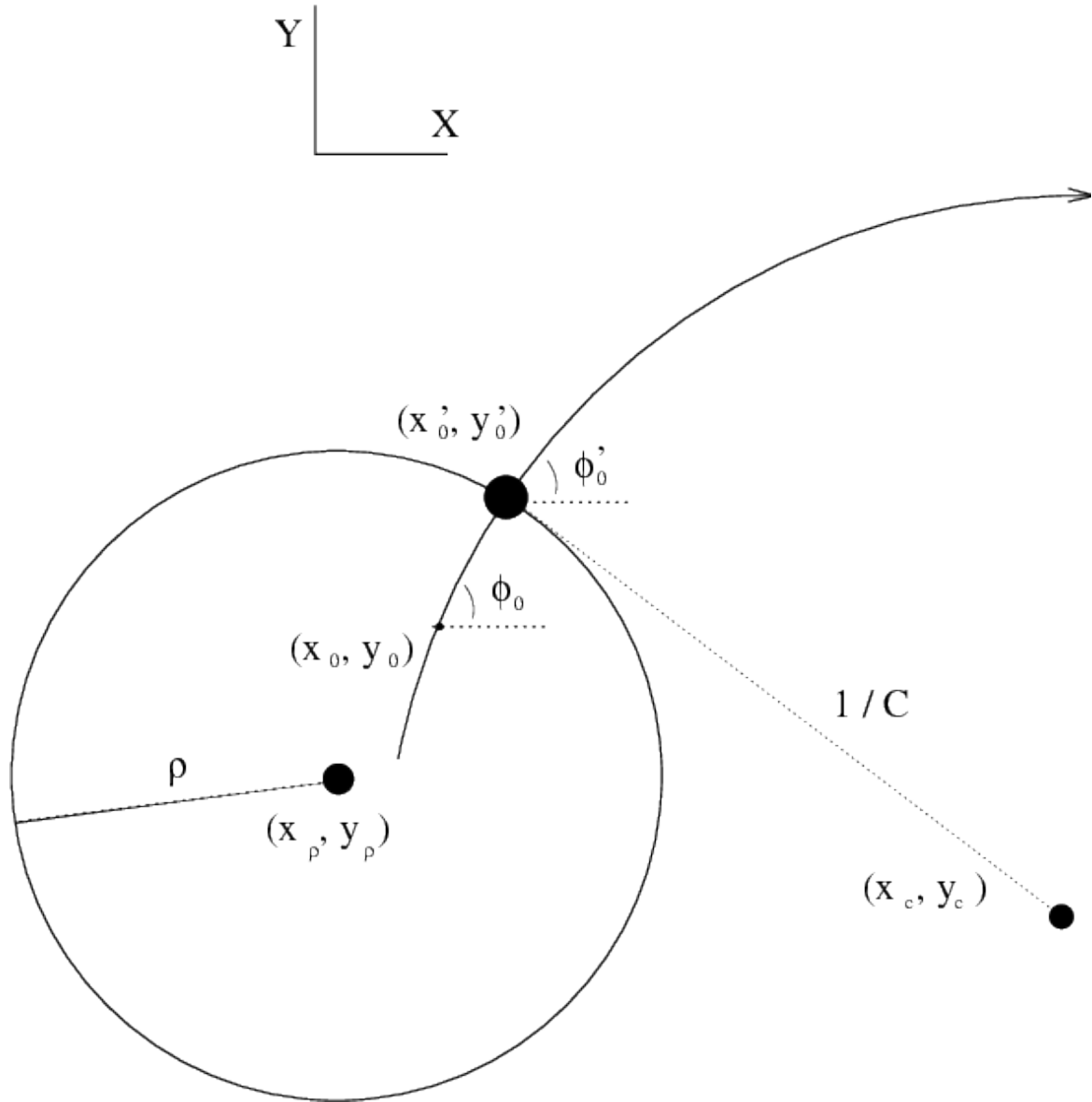


Figure 4. XY projection of a track crossing a cylinder with axis parallel to the Z axis.

In a typical scenario with a cylinder with $\rho^2 < R^2$ and $(x_0 - x_\rho)^2 + (y_0 - y_\rho)^2 < \rho^2$, i.e. a cylinder with its center close to the distance of minimum approach, only one solution with positive arc length exists, given by:

$$\phi_c = \phi_\rho + \arcsin(\gamma) \quad (\text{Eq. 60})$$

The covariance matrix at the new crossing point can be determined from the equations developed in Section 4, using (x'_0, y'_0, z'_0) as the new reference point of the trajectory. Note that, by construction, the reference point is sitting on the trajectory and therefore $\delta' = 0$.

4.5 TRACK CROSSING A PLANE

This case is illustrated in Figure 5. The crossing point (x'_0, y'_0, z'_0) can be found from the following conditions:

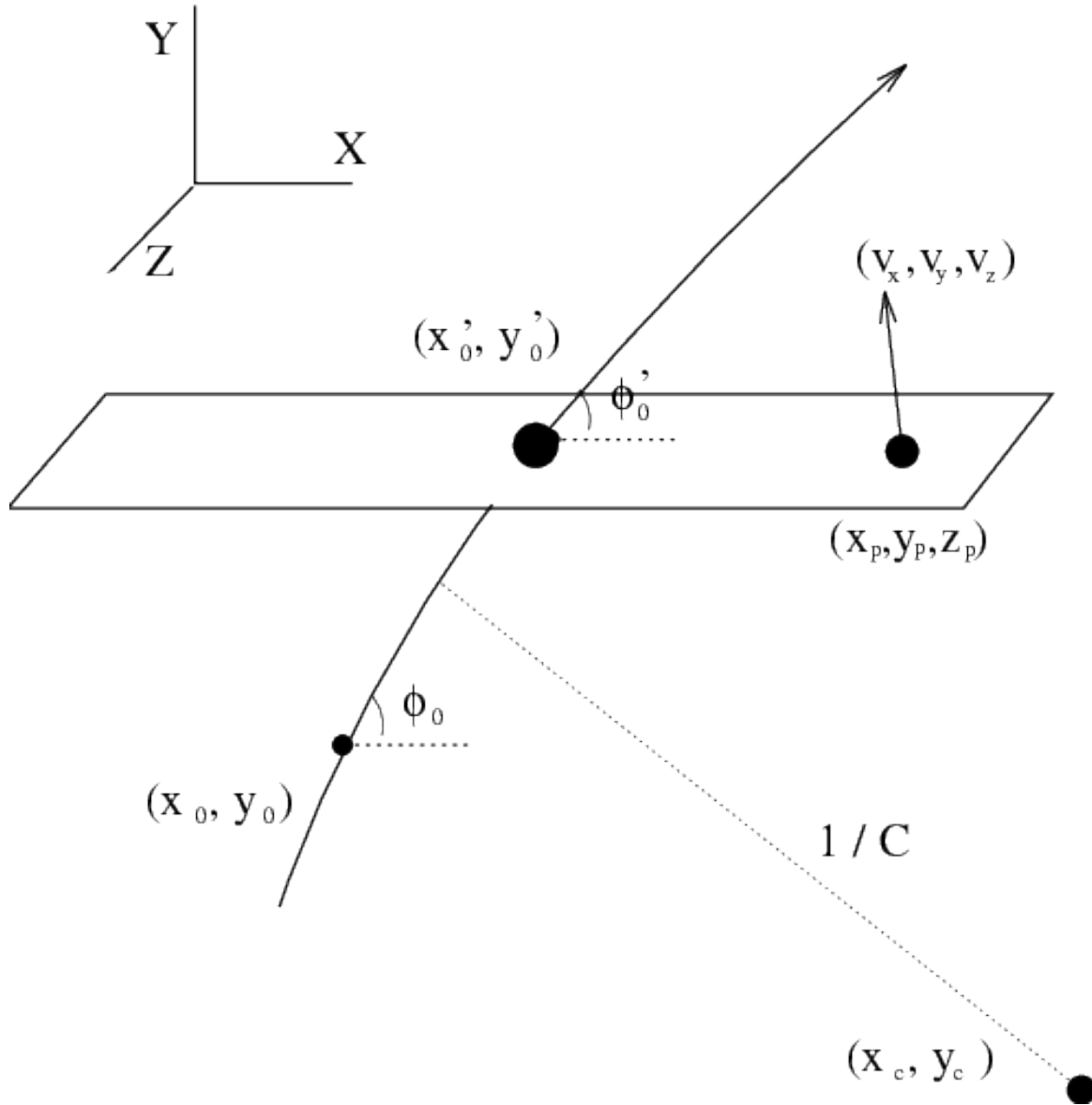


Figure 5. Parameters relevant to find the point where a track crosses a plane. In the figure (x_p, y_p, z_p) is a point in the plane and (v_x, v_y, v_z) is a unitary vector perpendicular to it.

$$x'_0 = x_0 + s \operatorname{sinc} \left(\frac{Cs}{2} \right) \cos \left(\phi_0 - \frac{Cs}{2} \right) \quad (\text{Eq. 61})$$

$$y'_0 = y_0 + s \operatorname{sinc} \left(\frac{Cs}{2} \right) \sin \left(\phi_0 - \frac{Cs}{2} \right) \quad (\text{Eq. 62})$$

$$z'_0 = z_0 + s \tan \lambda \quad (\text{Eq. 63})$$

$$0 = (x_p - x'_0) v_x + (y_p - y'_0) v_y + (z_p - z'_0) v_z \quad (\text{Eq. 64})$$

where (x_p, y_p, z_p) is a point in the plane and (v_x, v_y, v_z) is a unitary vector perpendicular to it. To solve the problem we define:

$$\phi_v = \text{atan2}(v_y, v_x) \quad (\text{Eq. 65})$$

$$v_t = \sqrt{v_x^2 + v_y^2} \quad (\text{Eq. 66})$$

$$d_p = (x_p - x_0) v_x + (y_p - y_0) v_y + (z_p - z_0) v_z \quad (\text{Eq. 67})$$

where d_p is actually the signed distance from (x_0, y_0, z_0) to the plane. The solution in terms of the variable s can be obtained from following equation:

$$s = \frac{d_p}{v_t \operatorname{sinc}\left(\frac{Cs}{2}\right) \cos\left(\frac{Cs}{2} + \phi_v - \phi_0\right) + v_z \tan \lambda} \quad (\text{Eq. 68})$$

which makes sense only if $|Cs| < \pi$. From a practical point of view it is convenient to follow an iterative method converging to the solution:

$$s_0 = \frac{d_p}{v_t \cos(\phi_v - \phi_0) + v_z \tan \lambda} \quad (\text{Eq. 69})$$

$$s_1 = \frac{d_p}{v_t \operatorname{sinc}\left(\frac{Cs_0}{2}\right) \cos\left(\frac{Cs_0}{2} + \phi_v - \phi_0\right) + v_z \tan \lambda} \quad (\text{Eq. 70})$$

⋮

$$s_i = \frac{d_p}{v_t \operatorname{sinc}\left(\frac{Cs_{i-1}}{2}\right) \cos\left(\frac{Cs_{i-1}}{2} + \phi_v - \phi_0\right) + v_z \tan \lambda} \quad (\text{Eq. 71})$$

Explicit solutions exist when $v_t = 0$ or $v_z = 0$. For $v_t = 0$ (plane perpendicular to the Z axis):

$$s = \frac{d_p}{v_z \tan \lambda}, \quad (\text{Eq. 72})$$

whereas for the $v_z = 0$ case (plane parallel to the Z axis):

$$\sin(Cs + \phi_v - \phi_0) = \sin(\phi_v - \phi_0) + C d_p, \quad (\text{Eq. 73})$$

which gives two possible solutions for s . Only the one with minimum arc length will be of interest in general. It is advisable to determine the final s value from Cs not by division, but by substitution in Equation Eq. 68, in order to keep the best possible accuracy.

5 INCLUDING NEW POINTS IN A TRACK

Let us consider a new measurement $(x_{meas}, y_{meas}, z_{meas})$ with some associated uncertainty. If the original track parameters and covariance matrix are known, it is always possible to improve the parameters of a track via χ^2 methods. According to the approach of Section 4, we will also assume that new measurement can be decomposed into uncorrelated measurements in the XY and SZ planes, such that XY and SZ fitting problems can be treated separately.

5.1 XY PLANE

Let us first redefine the track parameters using the new measurement $(x_{meas}, y_{meas}, z_{meas})$ as a new reference point. The new constraint can be interpreted as: $\delta = 0 \pm \sigma$, where σ is the transverse uncertainty on the measured position. The following χ^2 can then be defined:

$$\chi^2 = x_i S_{ij}^0 x_j + \left(\frac{\delta}{\sigma}\right)^2 = x_i S_{ij}^0 x_j + \left(\frac{\delta^0 + x_3}{\sigma}\right)^2 \quad (\text{Eq. 74})$$

where $x_1 = C - C^0$, $x_2 = \phi_0 - \phi_0^0$, $x_3 = \delta - \delta^0$, and a sum on repeated indices is assumed. S_{ij}^0 is the inverse of the original covariance matrix, V_{ij}^0 , and C^0 , ϕ_0^0 and δ^0 are the original values of the track parameters extrapolated to the new reference point. The minimum of this χ^2 corresponds to:

$$V_{kj} = \left(S_{kj}^0 + \frac{1}{\sigma^2} g_{k3} g_{j3}\right)^{-1} \quad (\text{Eq. 75})$$

↓

$$C = C^0 - V_{13} \frac{\delta^0}{\sigma^2} \quad (\text{Eq. 76})$$

$$\phi_0 = \phi_0^0 - V_{23} \frac{\delta^0}{\sigma^2} \quad (\text{Eq. 77})$$

$$\delta = \delta^0 - V_{33} \frac{\delta^0}{\sigma^2} \quad (\text{Eq. 78})$$

where g_{ij} is the identity matrix (1 if $i = j$, 0 otherwise).

5.2 SZ PLANE

The solution in this case is:

$$W_{ij} = \left(\begin{array}{cc} T_{11}^0 + \frac{s^2}{\sigma^2} & T_{12}^0 + \frac{s}{\sigma^2} \\ T_{21}^0 + \frac{s}{\sigma^2} & T_{22}^0 + \frac{1}{\sigma^2} \end{array} \right)^{-1} \quad (\text{Eq. 79})$$

↓

$$\tan \lambda = \tan \lambda^0 - (W_{11} s + W_{12}) \frac{z_0^0 + s \tan \lambda - z_{meas}}{\sigma^2} \quad (\text{Eq. 80})$$

$$z_0 = z_0^0 - (W_{21} s + W_{22}) \frac{z_0^0 + s \tan \lambda - z_{meas}}{\sigma^2} \quad (\text{Eq. 81})$$

where $\tan \lambda^0$ and z_0^0 are the original SZ parameters of the track, σ_z is the uncertainty on z_{meas} , s is the arc length described in the XY plane and T_{ij}^0 is the inverse of W_{ij}^0 , the original covariance matrix in the SZ plane.

5.3 CONSTRAINING A TRACK TO A POINT

Sometimes we are not interested in a precise estimate of the uncertainties when a new measurement is added to a track, but only in the improvements obtained in the limit in which this measurement is much more precise than the uncertainty of the track extrapolation to it, i.e. the limit in which a track is constrained to a point. We will denote the new measurement by $(x_{fix}, y_{fix}, z_{fix})$. In this case the equations presented in the previous section can be largely simplified.

In the system in which the fixed point is the reference point, the solution for the XY plane when $\sigma \rightarrow 0$ is:

$$C = C^0 - \frac{V_{13}^0}{V_{33}^0} \delta^0 \quad (\text{Eq. 82})$$

$$\phi_0 = \phi_0^0 - \frac{V_{23}^0}{V_{33}^0} \delta^0 \quad (\text{Eq. 83})$$

$$\delta = 0 \quad (\text{Eq. 84})$$

and a convenient parametrization of the covariance matrix in this limit is:

$$V_{ij} = \begin{pmatrix} \frac{S_{22}^0}{S_{11}^0 S_{22}^0 - S_{12}^0 S_{12}^0} & \frac{-S_{12}^0}{S_{11}^0 S_{22}^0 - S_{12}^0 S_{12}^0} & 0 \\ \frac{-S_{12}^0}{S_{11}^0 S_{22}^0 - S_{12}^0 S_{12}^0} & \frac{S_{11}^0}{S_{11}^0 S_{22}^0 - S_{12}^0 S_{12}^0} & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} \quad (\text{Eq. 85})$$

where the true (but small) value of the uncertainty σ must be used to ensure that the covariance matrix is not singular. In the SZ plane we obtain:

$$\tan \lambda = \tan \lambda^0 - \frac{sW_{11}^0 + W_{12}^0}{s^2W_{11}^0 + 2sW_{12}^0 + W_{22}^0} (z_0^0 + s \tan \lambda^0 - z_{fix}) \quad (\text{Eq. 86})$$

$$z_0 = z_{fix} - s \tan \lambda \quad (\text{Eq. 87})$$

with a covariance matrix in the $\sigma_z \rightarrow 0$ limit given by:

$$W_{ij} = \frac{1}{T_{11}^0 - 2sT_{12}^0 + s^2T_{22}^0} \begin{pmatrix} 1 + T_{22}^0 \sigma_z^2, & -s - T_{12}^0 \sigma_z^2 \\ -s - T_{12}^0 \sigma_z^2, & s^2 + T_{11}^0 \sigma_z^2 \end{pmatrix} \quad (\text{Eq. 88})$$

where again a non-null value of σ_z is necessary to avoid potential singularities.

6 COMMON VERTEX FOR SEVERAL TRACKS IN XY

Finding the common vertex for several tracks (at least 2) in the XY plane is equivalent to finding a new common reference point, (x'_r, y'_r) that globally minimizes the distances of closest approach of all those tracks. We address the problem using Equation Eq. 20 as starting point:

$$\delta'_i = \delta_i - \frac{C_i \delta_i^2}{2} + (\Delta_x \sin \phi_{0i} - \Delta_y \cos \phi_{0i}) (1 - C_i \delta_i) - C_i \frac{\Delta_x^2 + \Delta_y^2}{2} \quad (\text{Eq. 89})$$

where δ'_i is the distance of closest approach of the i th track to the common vertex (x'_r, y'_r) . For convenience we choose to determine the parameters Δ_x and Δ_y , which are the coordinates of the vector that connects the initial and final reference points: $(\Delta_x, \Delta_y) \equiv (x'_r - x_r, y'_r - y_r)$. Explicitly, the χ^2 to be minimized is:

$$\chi_{xy}^2 = \sum_{i=1}^N \left(\frac{\delta'_i}{\sigma_i} \right)^2 \quad (\text{Eq. 90})$$

where σ_i is the error the distance of closest approach for the i th track.

The safest way to find the solution in terms of Δ_x and Δ_y the use standard minimization programs like Minuit [14]. Nevertheless, if the initial reference point (x_r, y_r) is expected to be close to the final vertex, the solution can be found by iteration using a Newton minimization method. The method is expected to converge as long as $C_i (\Delta_x^2 + \Delta_y^2)$ terms stay small in the process. This is typically the case of the determination of the primary vertex of the event, where the initial reference can be well approximated by a nominal collision vertex position and the number of tracks is sufficiently large to avoid the presence of far minima. Expanding the χ^2 up to terms quadratic in Δ_x, Δ_y :

$$\alpha_i = \frac{2 \delta_i - C_i \delta_i^2}{2 \sigma_i} \quad (\text{Eq. 91})$$

$$\beta_i = \frac{\sin \phi_{0i} (1 - C_i \delta_i)}{\sigma_i} \quad (\text{Eq. 92})$$

$$\gamma_i = -\frac{\cos \phi_{0i} (1 - C_i \delta_i)}{\sigma_i} \quad (\text{Eq. 93})$$

↓

$$\chi_{xy}^2 = \sum_{i=1}^N \left[\alpha_i^2 + \begin{pmatrix} 2\alpha_i\beta_i & 2\alpha_i\gamma_i \end{pmatrix} \begin{pmatrix} \Delta_x \\ \Delta_y \end{pmatrix} \right] \quad (\text{Eq. 94})$$

$$+ \begin{pmatrix} \Delta_x & \Delta_y \end{pmatrix} \begin{pmatrix} \beta_i^2 - C_i \alpha_i & \beta_i \gamma_i \\ \beta_i \gamma_i & \gamma_i^2 - C_i \alpha_i \end{pmatrix} \begin{pmatrix} \Delta_x \\ \Delta_y \end{pmatrix} + \dots \quad (\text{Eq. 95})$$

The minimization of the quadratically truncated χ^2 leads to the solution:

$$\begin{pmatrix} \Delta_x \\ \Delta_y \end{pmatrix} = - \begin{pmatrix} \beta_i^2 - C_i \alpha_i & \beta_i \gamma_i \\ \beta_i \gamma_i & \gamma_i^2 - C_i \alpha_i \end{pmatrix}^{-1} \begin{pmatrix} \sum \alpha_i \beta_i \\ \sum \alpha_i \gamma_i \end{pmatrix} \quad (\text{Eq. 96})$$

This Newton step must be iterated until the vertex converges to a stable value. The minimization can be stopped for instance when the distance between the vertices of two consecutive iterations becomes of the order of the average impact parameter uncertainty: $\approx (\sum 1/\sigma_i^2)^{-1/2}$. Note that, in each iteration, the reference point is the vertex estimated in the previous iteration. Accordingly, the parameters ϕ_{0i} and δ_i will have to be propagated to the new reference point using Equations Eq. 15 and Eq. 16 before the iteration starts. Let us point out that, if convenient, the parameters of the tracks used in the vertex determination can be further improved a posteriori by constraining them to the common vertex ³, using the equations developed in Section 5.

³In principle there is a remaining correlation effect between each track parameter and the common vertex position, which should be taken into account. However, this correlation is expected to be small as long as the total number of tracks is sufficiently large.

7 COMMON VERTEX FOR SEVERAL TRACKS IN SPACE

The χ^2 contribution in the SZ plane is:

$$\chi_z^2 = \sum_{i=1}^N \left(\frac{\Delta_Z - z_{0i} - \frac{\tan \lambda_i}{\text{sinc}(C_i s_i)} (\Delta_x \cos \phi_{0i} + \Delta_y \sin \phi_{0i})}{\sigma_{zi}} \right)^2 \quad (\text{Eq. 97})$$

where s_i is the arc length from a displacement given by (Δ_x, Δ_y) and σ_{zi} is the uncertainty on z_{0i} . Note that, due to the convention used in the parametrization, Δ_Z is not a distance with respect to the reference point, but with respect to $z = 0$. Again, $C\delta'$ terms are neglected. The global χ^2 to be minimized is:

$$\chi^2 = \chi_{xy}^2 + \chi_{sz}^2, \quad (\text{Eq. 98})$$

which can be easily solved for $(\Delta_x, \Delta_y, \Delta_z)$ using standard minimization programs. Similarly to the XY case, one can also try a Newton minimization procedure if the common vertex is not expected to be far from the initial reference point. In this 3D case, the solution of the quadratically truncated χ^2 at each minimization step can be obtained from the following definitions:

$$\alpha_i = \frac{2\delta_i - C_i \delta_i^2}{2\sigma_i} \quad (\text{Eq. 99})$$

$$\beta_i = \frac{\sin \phi_{0i} (1 - C_i \delta_i)}{\sigma_i} \quad (\text{Eq. 100})$$

$$\gamma_i = -\frac{\cos \phi_{0i} (1 - C_i \delta_i)}{\sigma_i} \quad (\text{Eq. 101})$$

$$\xi_i = \frac{\cos \phi_{0i} \tan \lambda_i}{\text{sinc}(C_i s_i) \sigma_{zi}} \quad (\text{Eq. 102})$$

$$\eta_i = \frac{\sin \phi_{0i} \tan \lambda_i}{\text{sinc}(C_i s_i) \sigma_{zi}} \quad (\text{Eq. 103})$$

$$W_{\Delta_x \Delta_y \Delta_z} = \begin{pmatrix} \sum(\beta_i^2 + \xi_i^2 - C_i \alpha_i) & \sum(\beta_i \gamma_i + \xi_i \eta_i) & -\sum \frac{\xi_i}{\sigma_{zi}} \\ \sum(\beta_i \gamma_i + \xi_i \eta_i) & \sum(\gamma_i^2 + \eta_i^2 - C_i \alpha_i) & -\sum \frac{\eta_i}{\sigma_{zi}} \\ -\sum \frac{\xi_i}{\sigma_{zi}} & -\sum \frac{\eta_i}{\sigma_{zi}} & \sum \frac{1}{\sigma_{zi}^2} \end{pmatrix}^{-1}, \quad (\text{Eq. 104})$$

leading to the solution:

$$\begin{pmatrix} \Delta_x \\ \Delta_y \\ \Delta_z \end{pmatrix} = -W_{\Delta_x \Delta_y \Delta_z} \begin{pmatrix} \sum (\alpha_i \beta_i + z_{0i} \frac{\xi_i}{\sigma_{z_i}}) \\ \sum (\alpha_i \gamma_i + z_{0i} \frac{\eta_i}{\sigma_{z_i}}) \\ - \sum \frac{z_{0i}}{\sigma_{z_i}^2} \end{pmatrix} \quad (\text{Eq. 105})$$

Beside ϕ_{0i} and δ_i , now also s_i will have to be propagated to the estimated vertex from χ^2 truncation before starting a new iteration. This can be done using Equations Eq. 9 or Eq. 10.

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APPENDIX A. ALTERNATIVE TRACK PARAMETRIZATIONS.

Here we collect the Jacobians $J \left(\frac{\partial(\text{new parameters})}{\partial(C, \phi_0, \delta, \tan \lambda, z_0)} \right)$ for the alternative parametrizations discussed in subsection 2.1:

- Perigee parametrization [9]:

$$\rho \equiv -C \quad (\text{Eq. 106})$$

$$\phi_p \equiv \phi_0 \quad (\text{Eq. 107})$$

$$\epsilon \equiv -\delta \quad (\text{Eq. 108})$$

$$\theta \equiv \frac{\pi}{2} - \lambda \quad (\text{Eq. 109})$$

$$z_p \equiv z_0 \quad (\text{Eq. 110})$$

$$J \left(\frac{\partial(\rho, \phi_p, \epsilon, \theta, z_p)}{\partial(C, \phi_0, \delta, \tan \lambda, z_0)} \right) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\cos^2 \lambda & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{Eq. 111})$$

- Modified perigee parametrization [10]:

$$\frac{q}{p} \equiv \frac{C \cos \lambda}{0.29979 B} \quad (\text{Eq. 112})$$

$$\phi_p \equiv \phi_0 \quad (\text{Eq. 113})$$

$$\epsilon \equiv -\delta \quad (\text{Eq. 114})$$

$$\theta \equiv \frac{\pi}{2} - \lambda \quad (\text{Eq. 115})$$

$$z_p \equiv z_0 \quad (\text{Eq. 116})$$

$$J \left(\frac{\partial(q/p, \phi_p, \epsilon, \theta, z_p)}{\partial(C, \phi_0, \delta, \tan \lambda, z_0)} \right) = \begin{pmatrix} \frac{\cos \lambda}{0.29979 B} & 0 & 0 & -\frac{C \sin \lambda \cos^2 \lambda}{0.29979 B} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\cos^2 \lambda & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{Eq. 117})$$

- Curvilinear parametrization at the minimum distance of transverse approach [12]:

$$\frac{q}{p} \equiv \frac{C \cos \lambda}{0.29979 B} \quad (\text{Eq. 118})$$

$$\phi \equiv \phi_0 \quad (\text{Eq. 119})$$

$$x_{\perp} \equiv \delta \quad (\text{Eq. 120})$$

$$\lambda \quad \text{unchanged} \quad (\text{Eq. 121})$$

$$z_{\perp} \equiv z_0 \cos \lambda \quad (\text{Eq. 122})$$

$$J \left(\frac{\partial(q/p, \phi, x_{\perp}, \lambda, z_{\perp})}{\partial(C, \phi_0, \delta, \tan \lambda, z_0)} \right) = \left(\begin{array}{c|c|c|c|c} \frac{\cos \lambda}{0.29979 B} & 0 & 0 & -\frac{C \sin \lambda \cos^2 \lambda}{0.29979 B} & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -\cos^2 \lambda & 0 \\ \hline 0 & 0 & 0 & -z_0 \sin \lambda \cos^2 \lambda & \cos \lambda \end{array} \right) \quad (\text{Eq. 123})$$

